Sliding Abrikosov vortex lattice in the presence of a regular array of columnar pinning centers: ac conductivity and criticality near the transition to a pinned state

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The dynamics of the flux lattice in the mixed state of strongly type-II superconductor near the upper critical field $H_{c2}(T)$ subjected to ac field and interacting with a periodic array of short-range pinning centers (nanosolid) is considered. The superconductor in a magnetic field in the absence of thermal fluctuations on the mesoscopic scale is described by the time-dependent Ginzburg-Landau equations. An exact expression for the ac resistivity in the case of a δ-function model for the pinning centers in which the nanosolid is commensurate with the Abrikosov lattice (vortices outnumber pinning centers) is obtained. It is found that below a certain critical pinning strength u_c and sufficiently low frequencies there exists a sliding Abrikosov lattice, which moves nearly uniformly despite interactions with the pinning centers. At small frequencies the conductivity diverges as $(u-u_c)^{-1}$, whereas the ac conductivity on the depinning line diverges as $i\omega^{-1}$. This sliding lattice behavior, which does not exists in the single vortex-pinning regime, becomes possible due to strong interactions between vortices when they outnumber the columnar defects. Physically it is caused by "liberation" of the temporarily trapped vortices by their freely moving neighbors.

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I. INTRODUCTION

The great interest in the problem of magnetic-flux pinning in type-II superconductors is associated with its relevance to technological applications of superconductivity as well as with its implications to the general problem of complex glass dynamics with tunable parameters. An important challenge in applications of type-II superconductors is in achieving optimal critical currents under given magnetic fields. This requires preventing depinning of Abrikosov vortices during formation of the resistive state under the applied current. Random pointlike pinning centers naturally appear due to imperfections of lattice structure or chemical disorder. Pinning can be artificially enhanced by ion irradiation,[1] in which case one obtains a *random* array of columnar defects. Theory of dynamics of the pinned vortex matter by a random distribution of pins is very complicated.[2–4] However in the absence of significant thermal fluctuations on the mesoscopic scale the problem simplifies considerably. It was studied theoretically mostly in two-dimensional (2D) systems using either numerical methods within a model of interacting points such as particles representing vortices subject to pinning potential and driving force[5] or within the elasticity theory[3] in which the vortex matter is treated as an elastic manifold subject to both pinning stress and driving force. Elastic manifold under and ac electromagnetic radiation was investigated in Ref. 6.

Recently there have been advances in the study of vortex pinning by fabricating periodic arrays of pinning sites where each pinning site may be either magnetic or normal nanopattern inclusion effectively trapping vortices. Pinning arrays with triangular, square, and rectangular geometries have been fabricated using either microholes or blind holes,[7] arrays of magnetic dots,[8] and periodic array of columnar defects.[9] The resulting critical current is enhanced when vortex lattice is commensurate with the periodic array of pinning sites.[8] In addition this system is a convenient experimental tool to study the general problem of interacting periodic system moving in periodical potential such as dislocations in crystals or charge-density waves.[10] Theory of the Abrikosov lattice subjected to an ac field and periodic pinning is simpler but so far has been treated either numerically using molecular-dynamics approach[11] or by means of the elastic manifold approach.[12] In the one-dimensional situation of the vortex transport in narrow channels a sliding vortex phase was studied numerically[13] following the Frenkel-Kontorova model approach.[14]

Theoretically a basic question concerns the importance of correlation of defect centers for pinning of highly correlated vortex matter. It is sometimes believed that pinning of vortex lines in type-II superconductors is analogous to localization of correlated electrons by impurities in metals and semiconductors. According to this line of thought the vortices are mapped onto quantum particles, rather than considered as classical line—such as objects. If this were the case, a periodic array would not be able to trap the vortex lattice. However the vortices are topological solitons of the essentially nonlinear Ginzburg-Landau equations and behave like classical objects. Consequently both the random pinning and highly correlated pinning are expected to result in a roughly similar critical current and other characteristics of the pinned state especially when the density of columnar defects is much smaller than the density of vortices. We therefore concentrate on a simpler problem of a periodic array of short-range pins (to be named "nanosolid") and employ the time-dependent Ginzburg-Landau (TDGL) equations for the order parameter Ψ (Ref. 15) to describe the dynamics of the vortex matter in a magnetic field.

Considering the electric current applied to the system as a perturbation, a linear-response theory is used allowing the calculation of the ac resistivity. In the absence of thermal fluctuations on the mesoscopic scale, an exact solution for the linear response in the case of a δ-function model for the pinning centers in which the nanosolid is commensurate with the Abrikosov lattice (vortices outnumber pinning centers) is obtained. In the strong magnetic field regime near $H_{c2}(T)$ investigated here, two features emerge as compared to the low-field regime, commonly studied previously by the London approach. The number of vortices [described by a set of zeros of the order parameter field $\Psi(x,y)$] significantly exceeds the number of columnar pins. In addition the customary London approach is inapplicable since the distances between vortices are not much larger than the size of the vortex cores. Interactions between vortices in this regime are significant and neighbors of the trapped vortices can liberate them from the potential trap resulting in a significant decrease in the critical current.

We find, that below a certain critical pinning strength u_c, Abrikosov lattice of vortices is moving coherently despite interactions with the pinning centers, thus forming a "sliding Abrikosov lattice state." The dependence of the ac conductivity on pinning strength u, magnetic induction and frequency in this phase is calculated. In particular, at small frequencies conductivity as a function of u behaves as $(u-u_c)^{-1}$, demonstrating an ideal metal behavior, $i\omega^{-1}$, as function of frequency at $u=u_c$.

The rest of the paper is organized as follows. In the next section the model is presented. In Sec. III a general linear-response theory of a strongly type-II superconductor under a magnetic field H to external alternating electric current is constructed to leading order in the small expansion parameter $a_h=[1-T/T_c-H/H_{c2}(0)]/2$. This parameter describes deviation of H from the mean-field upper critical field $H_{c2}(T)$. The condensate part of conductivity is presented via Green's function (GF) of quantum charged particle subjected to both magnetic field and periodic potential. The exact Green's function for the corresponding linearized TDGL equations in the presence of periodic array of short-range potentials (commensurate with the vortex lattice) is calculated using the inversion of the Lippmann-Schwinger equations[16] in Sec. IV. The results for the ac conductivity are presented and analyzed in Secs. V and VI. Criticality at low frequencies is discussed in Sec. VI. Technical details are relegated to Appendices.

II. MODEL

Let us consider a type-II superconductor under a constant external magnetic field $\mathbf{H}$ parallel to a system of columnar defects directed along z axis and carrying electric current along the y axis, see Fig. 1. The columnar defects are located at points $\mathbf{r}_a$ [2D vectors $\mathbf{r}=(x,y)$ will be denoted by bold letters] and are assumed to be short range (on the scale of the coherence length ξ).

A. Basic equations

The static magnetic properties of the superconductor are described by the GL Gibbs energy as a functional of the order parameter Ψ and vector potential $\mathbf{A}$,

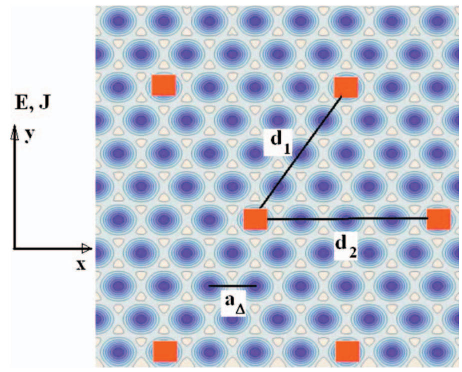

FIG. 1. (Color) Hexagonal Abrikosov vortex lattice (distribution of the superfluid density $|\psi(\mathbf{r})|^2$) and pinning centers. Zeros of order parameter fall on the locations of the columnar defects (red squares) so that vortices outnumber the pins. Vectors $\mathbf{d}_1$ and $\mathbf{d}_2$ are lattice vectors of the lattice of pinning sites. Distance between nearest neighbors of the Abrikosov lattice is a_Δ.

$$F_{\mathrm{GL}}[\Psi,\mathbf{A}]=\int dz d\mathbf{r}\left[\frac{\hbar^2}{2m_c^*}|\partial_z\Psi|^2+\frac{\hbar^2}{2m^*}|\mathbf{D}\Psi|^2-a'(\mathbf{r})|\Psi|^2+\frac{b'}{2}|\Psi|^4+\frac{1}{8\pi}(\mathbf{B}-\mathbf{H})^2\right]. \quad (1)$$

Here $\mathbf{D}\equiv\boldsymbol{\nabla}-i\frac{2\pi}{\Phi_0}\mathbf{A}$, denotes the covariant derivative and $\Phi_0=\frac{hc}{e^*}$, $e^*=2|e|$ is the unit of flux, $\mathbf{B}=\boldsymbol{\nabla}\times\mathbf{A}$ is the magnetic induction. We chose the vector potentials in the form independent of time,

$$A_x=-\frac{1}{2}By,\quad A_y=\frac{1}{2}Bx. \quad (2)$$

Assuming that the ratio $\kappa\equiv\lambda/\xi\gg1$, where λ is the penetration depth, the magnetization M is by the factor $1/\kappa^2$ smaller than the field and consequently for magnetic fields few times larger than H_{c1}, $B\approx H$. Thermal fluctuations on the mesoscopic scale are ignored. The temperature is taken into account only on the mean-field level, namely, coefficients of the Ginzburg-Landau energy depend on temperature T.

When the columnar defects are absent (the "clean" case), $a'(\mathbf{r})=\alpha(T_c-T)$ is uniform and the free energy is minimized by a hexagonal Abrikosov lattice of vortices with cores located at $\mathbf{r}_{n_1,n_2}=n_1\mathbf{a}_1+n_2\mathbf{a}_2$ with $\mathbf{a}_1=a_\Delta(1/2,\sqrt{3}/2)$, $\mathbf{a}_2=a_\Delta(1,0)$, where the lattice spacing is $a_\Delta=2^{1/2}3^{-1/4}\sqrt{\Phi_0/B}$. The columnar defects are represented by an inhomogeneous coefficient

$$a'(\mathbf{r})=\alpha[T_c-T+V(\mathbf{r})], \quad (3)$$

where V consists of "potentials" around pinning centers $\mathbf{r}_a$,

$$V(\mathbf{r})=T_c\sum_a U(\mathbf{r}-\mathbf{r}_a). \quad (4)$$

On microscopic scale the potential arises, for example, from deviation of local charge carriers density $n_e(\mathbf{r})$ from that of the uniform sample, n_0, sometimes represented as[2]

$$V(\mathbf{r}) = T_c \frac{\partial \ln T_c}{\partial \ln n_e} \frac{\delta n_e(\mathbf{r})}{n_0}. \tag{5}$$

We assume that the defects are thin on the scale of coherence length at certain temperature, which is quite large for low-T_c superconductors compared to the size of damage of ions or electrons used in irradiation experiments or nanosize antidots with effective radius w and strength $\varepsilon > 0$ considered as phenomenological parameters within the Ginzburg-Landau approach. As will be shown below, the only solvable configuration corresponds to a hexagonal periodic array of very thin pinning points located at $\mathbf{r}_a = n_1\mathbf{d}_1 + n_2\mathbf{d}_2$ commensurate with the static Abrikosov lattice so that $\mathbf{d}_1 = s_1\mathbf{a}_1$, $\mathbf{d}_2 = s_2\mathbf{a}_2$ with s_1, s_2 integers. The density of pinning centers is proportional to the fractional filling factor $f = s_1 s_2$, namely,

$$\frac{N_p}{L_x L_y} = \frac{2}{f\sqrt{3}a_\Delta^2}, \tag{6}$$

for hexagonal lattice. In particular, it means that magnetic field cannot be too small $B > \frac{\Phi_0 N_p}{L_x L_y}$.

The simplest relaxation dynamics of a superconductor in the presence of electric field is described by TDGL equation[15]

$$\frac{\hbar^2 \gamma}{2m^*} D_t \Psi = -\frac{\delta}{\delta \Psi^*} F_{\rm GL}, \tag{7}$$

where $D_t \equiv \frac{\partial}{\partial t} - i\frac{e^*}{\hbar}\Phi$ and the electric field is $\mathbf{E} = -\boldsymbol{\nabla}\Phi - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}$. The vortices are moving along the x direction, see Fig. 1. Maxwell equations are

$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J}, \quad \mathbf{J} = \mathbf{J}_n + \mathbf{J}_s. \tag{8}$$

Superconducting component of the current density is

$$\mathbf{J}_s = \frac{ie^*\hbar}{2m^*}(\Psi^* \mathbf{D}\Psi - \Psi \mathbf{D}\Psi^*), \tag{9}$$

while the normal electron component of the current density is $\mathbf{J}_n = \sigma_n \mathbf{E}$.

Neglecting the time dependence of the electric charge [screened on the Thomas-Fermi length, which is smaller than ξ (Ref. 15)], the charge conservation law in a superconductor reads

$$\nabla \cdot \mathbf{J} = 0. \tag{10}$$

In our case of large κ and the magnetic field not far from $H_{c2}(T)$, the charge conservation Eq. (10) implies homogeneity of the current density. Indeed taking curl of the Maxwell Eq. (8), one obtains

$$\nabla \times \mathbf{J} = \nabla^2 M \sim O(\kappa^{-2}). \tag{11}$$

Consequently the ac current density is uniform and is oriented along the y axis, $J(t) = J_0 \cos(\omega t)$.

B. Dimensionless units

The system is invariant under translations in the field directions, so we use a 2D dimensionless energy density $f_{\rm GL}$: $F_{\rm GL} = L_z H_{c2}^2/(8\pi\kappa^2)\int d\mathbf{r} f_{\rm GL}$. In this paper $\xi = \hbar/(2m^*\alpha T_c)^{1/2}$ will be used as a unit of length, $\mathbf{r} \to \mathbf{r}/\xi$, while $H_{c2} = \Phi_0/(2\pi\xi^2)$ is the unit of magnetic field, $h \equiv B/H_{c2}$. The scaled order parameter is defined by $\psi = 2^{-1/2}\Psi/\Psi_0$, where $\Psi_0 = (\alpha T_c/b')^{1/2}$, so that the dimensionless energy density is can be written in the following form

$$f_{\rm GL} = \psi^* \hat{H}\psi - a_h \psi^*\psi + \frac{1}{2}(\psi^*\psi)^2. \tag{12}$$

The dimensionless parameter,

$$a_h = \frac{1-t-h}{2} - u_0, \quad t = \frac{T}{T_c}, \tag{13}$$

has a physical meaning of "distance" from the static normal-mixed-state boundary in the $H-T$ space and the constant shift u_0 reflects an average pinning effect. It will be determined by exploiting the bifurcation (or transition) point expansion[17,18] around $a_h = 0$ at which the order parameter vanishes. Defining a Hamiltonian for the case without pinning,

$$\hat{H}_{cl} = -\frac{1}{2}D^2 - \frac{h}{2} \tag{14}$$

and a dimensionless pinning potential $U(\mathbf{r})$ one can represent the linear operator in Eq. (12) in the following form

$$\hat{H} = \hat{H}_{cl} - u_0 - \sum_a U(r - r_a) \equiv \hat{H}_p - u_0. \tag{15}$$

It should have the property[17,18] that its lowest eigenvalue is zero. It is well known that the lowest eigenstate of the Landau Hamiltonian H_{cl} is degenerate, however the presence of the pinning potential in Eq. (15) partially lifts the Landau degeneracy. The lowest energy corresponds to the lowest Landau-level (LLL) state in which zeros of the Abrikosov wave function fall on the pining center sites $\mathbf{r}_a$, see Fig. 1 and expression in Appendix A.

In analogy to the coherence length, one can define a characteristic time scale. In the superconducting phase that is a typical "relaxation" time is $t_{\rm GL} = \gamma\xi^2/2$ and unit of electric field, $E_{\rm GL} = H_{c2}\xi/(ct_{\rm GL})$, $\mathcal{E} = E/E_{\rm GL}$. In these units the time-dependent equation takes a form

$$-\partial_t \psi = \hat{H}\psi - a_h\psi + \psi^*\psi^2 - i\Phi\psi. \tag{16}$$

The current density is scaled as $\mathbf{J} = cH_{c2}/(2\pi\xi\kappa^2)j$, in particular,

$$\mathbf{j}_s = \frac{i}{2}[\psi^* \mathbf{D}\psi - \psi(\mathbf{D}\psi)^*] \tag{17}$$

being the dimensionless supercurrent density. The conductivity will be given in units of

$$\sigma_{\rm GL} = \frac{c^2 t_{\rm GL}}{2\pi\lambda^2} = \frac{c^2\gamma}{4\pi\kappa^2}. \tag{18}$$

This unit is close to the normal-state conductivity σ_n in low-T_c superconducting metals in the dirty limit, for which $\sigma_n = c^2\gamma/(8\pi\kappa^2)$.[15] In general there is a factor k of order 1 relating the two: $\sigma_n = k\sigma_{GL}$. The corresponding equation for the dimensionless electric field, takes a form

$$k\mathcal{E} = j - j_s. \tag{19}$$

From now on we will take for simplicity $k=1$. The model will be solved using expansions in electric field and a_h but exactly in pinning strength in the next section.

III. LINEAR-RESPONSE THEORY AND EXPANSION IN POWERS OF SUPERFLUID DENSITY (PARAMETER a_h)

A. Linear response of the system to electric current

The set of Eqs. (16) and (19) can be solved using expansion in both the total current, which can be viewed as "external," and the parameter a_h defined in Eq. (13). To the first order in j the order parameter is represented as follows:

$$\psi(\mathbf{r},t) = \phi(\mathbf{r}) + \theta(\mathbf{r},t), \tag{20}$$

where ϕ is the static mean-field solution specified below. Substituting Eq. (20) into TDGL, Eq. (16), one obtains

$$-\partial_t(\phi+\theta) = \hat{H}(\phi+\theta) - a_h(\phi+\theta) + (\phi+\theta)^*(\phi+\theta)^2 - i\Phi\phi. \tag{21}$$

To this order in j one observes that the correction to the order parameter, θ, satisfies

$$\partial_t\theta = (-\hat{H} + a_h - 2\phi^*\phi)\theta - \theta^*\phi^2 + i\Phi\phi. \tag{22}$$

Similarly substituting Eq. (20) into the expression for y component of the supercurrent Eq. (17), Eq. (19) takes a form

$$\mathcal{E} = j - i[\theta^* D_y\phi - \theta(D_y\phi)^*]. \tag{23}$$

This equation expresses the linear response since the correction θ in turn depends on the current via the electric potential Φ appearing in Eq. (22). The equation Eq. (22) can be solved using expansion in the small parameter a_h.

B. Expansion in a_h

It is important to note that by exploiting the ultimate localized form, i.e., proportional to a sum of delta functions, for the pinning potential $U(\mathbf{r})$ in Eq. (4), the static configuration, $\phi(\mathbf{r})$, of the order parameter to leading order in a_h, can be calculated exactly. In fact, the definition of the parameter a_h in Eq. (13) already took this fact into account. Indeed, the solution at small a_h without pinning is well known[17,18]

$$\phi(\mathbf{r}) = \left(\frac{a_h}{\beta_A}\right)^{1/2}\varphi_0(\mathbf{r}) + O(a_h^{3/2}) \tag{24}$$

with the functions $\varphi_N(\mathbf{r})$ constituting a basis of orbitals with Landau-level index $N=0,1,\ldots,$ defined in symmetric gauge in Appendix A. The next to leading order in a_h provides normalization and fixes certain linear combination of the LLL functions so that $\varphi_0(\mathbf{r})$ is a hexagonal lattice. Yet without pinning this solution is highly degenerate since one can shift or rotate the lattice as a whole.

When the commensurate pinning potential is added, the degeneracy is lifted and the only configuration of the minimal energy is the one with vortex cores located exactly at the sites of the columnar pins. Energy of all the other configurations are higher. The lowest eigenvalue of the linear operator $\hat{H}_p$, Eq. (15), [determining the $H_{c2}(T)$ line] is shifted by u_0. This justifies the dependence on the pinning strength in the definition of a_h, Eq. (13). It is therefore concluded that such a pinning does not change the shape of the configuration in the leading order but does suppress the value of the amplitude of the order parameter and the critical field.

Now we return to the linear-response relation, namely, to expansion of physical quantities to the first order in j, Eq. (23). It is clear in view of Eq. (24), that the correction to the order parameter θ is of order $a_h^{1/2}$ (thus allowing to neglect $a_h^{3/2}$ and higher order terms), and obeys

$$\partial_t\theta = -\hat{H}\theta + i\Phi\phi. \tag{25}$$

For the ac transport along the y direction

$$\Phi(r,t) = -\int_0^y \mathcal{E}(x,y',t)dy'. \tag{26}$$

Defining the retarded Green's function by

$$(\partial_t + \hat{H})G(r,r',t-t') = \delta(r-r',t-t'), \tag{27}$$

one writes the solution of Eq. (22),

$$\theta(r,t) = -i\int_{t'=-\infty}^{t}\int_{\mathbf{r}'} G(\mathbf{r},\mathbf{r}',t-t')\phi(\mathbf{r}')\int_0^{y'} dy''\mathcal{E}(x',y'',t'). \tag{28}$$

Since the supercurrent is on order of a_h, according to Eq. (23), electric field has an expansion: $\mathcal{E}=j+O(a_h)$. Using homogeneity of current density, the integral over y'' can be performed,

$$\theta(r,t) = -i\int_{t'=-\infty}^{t} j(t')\int_{\mathbf{r}'} G(\mathbf{r},\mathbf{r}',t-t')y'\phi(\mathbf{r}'). \tag{29}$$

Consequently the linear-response relation between the current density and the induced electric field, Eq. (23), can be written via the Green's function in the form

$$\mathcal{E}(\mathbf{r},t) = j(t) + \int_{t'=-\infty}^{t} j(t')\int_{\mathbf{r}'} G(\mathbf{r},\mathbf{r}',t-t')y'\phi(\mathbf{r}')(D_y\phi)^* + \text{c.c.} \tag{30}$$

Note that in contrast to the full current density $j(\mathbf{r},t)$, which is spatially uniform due to the charge conservation law, Eq. (10) and large κ, the electric field is spatially homogeneous only to leading order in the small parameter a_h. The inhomogeneous correction to the electric field in Eq. (30) (induced by the supercurrent j_s) is however responsible for the pinning mechanism investigated in the present paper.

C. ac resistivity

For a homogeneous ac current density $j(t)=j_0\cos(\omega t)$, Eq. (30), averaged over volume of the sample, gives the following expression for complex resistivity

$$\rho(\omega)=\frac{1}{T}\int_{t=0}^{T}e^{-i\omega t}\langle\mathcal{E}\rangle_{\mathbf{r}}/j_0, \tag{31}$$

where in the end the large time limit $T\to\infty$ should be taken. Performing integration over y'' one obtains (recalling that in our dimensionless units $\sigma_n=\rho_n=1$)

$$\rho(\omega)=1-\sigma_s(\omega)\approx\frac{1}{1+\sigma_s(\omega)}, \tag{32}$$

with the condensate contribution to conductivity

$$\sigma_s(\omega)=-\int_{r'}y'\langle\phi(\mathbf{r}')[D_y\phi(\mathbf{r})]^*G(\mathbf{r},\mathbf{r}',\omega)+\phi^*(\mathbf{r}')D_y\phi(\mathbf{r})G^*(\mathbf{r},\mathbf{r}',-\omega)\rangle_{\mathbf{r}}, \tag{33}$$

and the temporal Fourier transform was defined as $G(\mathbf{r},\mathbf{r}',t-t')=\frac{1}{2\pi}\int_\omega e^{-i\omega t}G(\mathbf{r},\mathbf{r}',\omega)$.

It is important to note that since the real part of the resistivity is positive, in view of Eq. (32), our theory strictly speaking is valid only when the condensate contribution to conductivity is smaller than σ_n. Equation (33) allows to relate the dynamic conductivity in the superconductor with the GF of a quantum-mechanical Hamiltonian $\hat{H}_p$ of a charged particle in magnetic field in the presence of a periodic potential defined in Eq. (15). This will also allow calculation of the lowest-energy eigenvalue determining the shift u. The next section and Appendix B deal with this problem.

IV. GREEN'S FUNCTION FOR A SYSTEM WITH PERIODIC δ-PINNING ARRAY

To find the GF we approximate the potential by an array of delta functions

$$U(\mathbf{r})=-U_0\sum_a\delta(\mathbf{r}-\mathbf{r}_a), \tag{34}$$

where

$$U_0=\frac{\pi w^2\varepsilon}{\xi^2T_c}, \tag{35}$$

and ε is the pinning energy. The delta function represents sufficiently localized defects on the scale of coherence length. First we review the comprehensively studied clean case.

A. Retarded Green's function G_{cl} for a system without pinning potential

Neglecting pinning, it can be easily seen that in the symmetric gauge

$$D_x=\frac{\partial}{\partial x}-i\frac{h}{2}y,\quad D_y=\frac{\partial}{\partial y}+i\frac{h}{2}x, \tag{36}$$

the function

$$G_{cl}(\mathbf{r},\mathbf{r}',t)=\exp\left[\frac{ih}{2}(xy'-yx')\right]g_{cl}(\mathbf{r}-\mathbf{r}',t) \tag{37}$$

satisfies Eq. (27) for the retarded GF for the operator $\hat{H}_{cl}$ defined in Eq. (15). Here

$$g_{cl}(\mathbf{r},t)=C(t)\exp\left[-\frac{r^2}{2\eta(t)}\right],$$

$$\eta(t)=\frac{2}{h}\tanh\left(\frac{ht}{2}\right),\quad C(t)=\frac{h}{4\pi}\exp\left(\frac{ht}{2}\right)\sinh^{-1}\left(\frac{ht}{2}\right) \tag{38}$$

is a gauge-invariant translation symmetric part of the GF. The Gaussian form of the GF allows analytic integration of the periodic pinning problem.

B. Relation between G_{cl} and Greens' function with periodic array of delta potentials

The GF with added potential $U(\mathbf{r})$, Eq. (34), obeys

$$[\partial_t+\hat{H}_p]G(\mathbf{r},\mathbf{r}',t-t')=\delta(\mathbf{r}-\mathbf{r}')\delta(t-t'), \tag{39}$$

where one chooses to divide the operator $\hat{H}_p$ of Eq. (15) into a solvable part $\hat{H}_{cl}$ related to the "clean case" of Sec. IV A and the delta potentials. Similar problems have been considered in quantum mechanics[16] using path integrals. We start from the related Lippmann-Schwinger equation linking the GF of $\hat{H}_{cl}$ and that of $\hat{H}_p$,

$$G(\mathbf{r},\mathbf{r}',t-t')=G_{cl}(\mathbf{r},\mathbf{r}',t-t')-\int d\mathbf{r}''\times\int dt''G_{cl}(\mathbf{r},\mathbf{r}'',t-t'')U(\mathbf{r}'')G(\mathbf{r}'',\mathbf{r}',t''-t'). \tag{40}$$

For the Fourier transform $G(\mathbf{r},\mathbf{r}',t-t')=\frac{1}{2\pi}\int_\omega e^{i\omega t}G(\mathbf{r},\mathbf{r}',\omega)$ the equation separates in frequencies

$$G(\mathbf{r},\mathbf{r}',\omega)=G_{cl}(\mathbf{r},\mathbf{r}',\omega)-\int d\mathbf{r}''G_{cl}(\mathbf{r},\mathbf{r}'',\omega)U(\mathbf{r}'')G(\mathbf{r}'',\mathbf{r}',\omega). \tag{41}$$

Substituting the potential Eq. (34) one obtains

$$G(\mathbf{r},\mathbf{r}',\omega)=G_{cl}(\mathbf{r},\mathbf{r}',\omega)-U_0\sum_a G_{cl}(\mathbf{r},\mathbf{r}_a,\omega)G(\mathbf{r}_a,\mathbf{r}',\omega). \tag{42}$$

In particular, at pinning points $\mathbf{r}=\mathbf{r}_b$ one gets [using Eq. (37)]

$$G(\mathbf{r}_b,\mathbf{r}',\omega)=G_{cl}(\mathbf{r}_b,\mathbf{r}',\omega)-U_0\sum_a e^{+ih/2\mathbf{r}_a\times\mathbf{r}_b}g_{cl}\times(\mathbf{r}_b-\mathbf{r}_a,\omega)G(\mathbf{r}_a,\mathbf{r}',\omega). \tag{43}$$

Here we assumed commensurability with vortex lattice for pins in zeros so there will be no phase factor on the right-hand side due to flux quantization,

$$\mathbf{r}_a \times \mathbf{r}_b = \frac{2\pi f}{h}(n_{a1}n_{b2} - n_{b1}n_{a2}). \quad (44)$$

If $f=s_1s_2$ is even the phase in Eq. (43) disappears while for an odd s_1s_2 one still can solve the pinning problem by dividing the pinning sites into two sublattices. We continue here with the even case. Under this conditions, translation symmetry allows to solve the set of linear equations

$$\sum_a M_{ba}(\omega)G(\mathbf{r}_a,\mathbf{r}',\omega) = G_{cl}(\mathbf{r}_b,\mathbf{r}',\omega) \quad (45)$$

with a symmetric matrix M defined by

$$M_{ba}(\omega) = \delta_{ab} + U_0 g_{cl}(\mathbf{r}_b - \mathbf{r}_a,\omega), \quad (46)$$

Using Fourier transform

$$g_{cl}(\mathbf{r}_a,\omega) = \frac{1}{(2\pi)^2}\int_{\mathbf{k}} g^{cl}_{\mathbf{k},\omega} \exp(i\mathbf{k}\cdot\mathbf{r}_a) = \frac{1}{(2\pi)^2}\int_{\mathbf{q}\in\mathrm{BZ}} \tilde{g}_{\mathbf{q},\omega} \exp(i\mathbf{q}\cdot\mathbf{r}_a). \quad (47)$$

The function $g_{cl}(\mathbf{r}_a,\omega)$ is more conveniently written via

$$\tilde{g}_{\mathbf{q},\omega} = \sum_{\mathbf{K}} g^{cl}_{(\mathbf{q}+\mathbf{K}),\omega}, \quad (48)$$

where $\mathbf{K}$ are reciprocal pinning centers lattice points

$$\mathbf{K} = m_1\mathbf{e}_1 + m_2\mathbf{e}_2,$$

$$\mathbf{e}_1 = \frac{2\pi}{s_1 a}\left(0,\frac{2}{\sqrt{3}}\right), \quad \mathbf{e}_2 = \frac{2\pi}{s_2 a}\left(1,-\frac{1}{\sqrt{3}}\right). \quad (49)$$

Here $a=a_\Delta/\xi=2\times 3^{-1/4}\sqrt{\pi/h}$ is the dimensionless Abrikosov lattice spacing.

The result reads formally

$$G(\mathbf{r}_a,\mathbf{r}',\omega) = \sum_b M^{-1}_{ab}(\omega)G_{cl}(\mathbf{r}_b,\mathbf{r}',\omega). \quad (50)$$

The Kronecker delta function appearing in Eq. (46) has the following integral representation,

$$\delta_{ab} = \frac{1}{S_{\mathrm{BZ}}}\int_{\mathbf{q}\in\mathrm{BZ}} e^{i\mathbf{q}\cdot(\mathbf{r}_b-\mathbf{r}_a)}, \quad (51)$$

where the area of the Brillouin zone (BZ) of the pinning centers lattice is

$$S_{\mathrm{BZ}} = (2\pi)^2\rho_p = \frac{2}{3^{1/2}f}\left(\frac{2\pi}{a}\right)^2 = \frac{2\pi h}{f}. \quad (52)$$

Using this representation the inverse matrix reads

$$M^{-1}_{ba}(\omega) = \frac{1}{S_{\mathrm{BZ}}}\int_{\mathbf{q}\in\mathrm{BZ}} e^{i\mathbf{q}\cdot(\mathbf{r}_b-\mathbf{r}_a)}\Pi_{\mathbf{q},\omega}, \quad (53)$$

where the polarization kernel is defined as

$$\Pi_{\mathbf{q},\omega} = \frac{1}{1+U_0\rho_p\tilde{g}_{\mathbf{q},\omega}}. \quad (54)$$

As a result Eq. (50) becomes explicit,

$$G(\mathbf{r}_a,\mathbf{r}',\omega) = \frac{1}{S_{\mathrm{BZ}}}\sum_b \int_{\mathbf{q}\in\mathrm{BZ}} e^{i\mathbf{q}\cdot(\mathbf{r}_b-\mathbf{r}_a)}\Pi_{\mathbf{q},\boldsymbol{\omega}}G_{cl}(\mathbf{r}_b,\mathbf{r}',\omega). \quad (55)$$

Substituting this into the expression of the full GF with arbitrary positions, Eq. (42), one obtains

$$G(\mathbf{r},\mathbf{r}',\omega) = G_{cl}(\mathbf{r},\mathbf{r}',\omega) - \frac{U_0}{S_{\mathrm{BZ}}}\sum_{a,b}\int_{\mathbf{q}} e^{i\mathbf{q}\cdot(\mathbf{r}_b-\mathbf{r}_a)}\Pi_{\mathbf{q},\boldsymbol{\omega}}G_{cl}(\mathbf{r},\mathbf{r}_a,\omega)G_{cl}(\mathbf{r}_b,\mathbf{r}',\omega). \quad (56)$$

The exact full GF allows to determine both the position of a new mean-field transition line ($a_h=0$) and the conductivity from Eq. (33).

C. Shift of the mean-field transition line

Position of the mean-field transition line is determined by the lowest eigenvalue of the operator $\hat{H}_p$, Eq. (15). This can be obtained from poles of the resolvent of the operator $\hat{H}_p$ which is simply related to the GF. In particular, the ground state is determined by the large-time asymptotic.

The resolvent of $\hat{H}_p$ is defined as

$$R(\omega) = \int_{\mathbf{r}} G(\mathbf{r},\mathbf{r},\omega). \quad (57)$$

Substituting Eq. (56) one obtains

$$R(\omega) = \int_{\mathbf{r}} G_{cl}(\mathbf{r},\mathbf{r},\omega) - \frac{U_0}{S_{\mathrm{BZ}}}\sum_{a,b}\int_{\mathbf{q}} e^{i\mathbf{q}\cdot(\mathbf{r}_b-\mathbf{r}_a)}\Pi_{\mathbf{q},\omega}\int_{\mathbf{r}} G_{cl}(\mathbf{r},\mathbf{r}_a,\omega)G_{cl}(\mathbf{r}_b,\mathbf{r},\omega). \quad (58)$$

It is shown in Appendix B that in our case integration over the small Brillouin zone can be approximated by taking $\mathbf{q}=0$ in the polarization kernel $\Pi_{\mathbf{q},\boldsymbol{\omega}}$

$$R(\omega) = \int_{\mathbf{r}} G_{cl}(\mathbf{r},\mathbf{r},\omega) - U_0\Pi_{\mathbf{q}=0,\boldsymbol{\omega}}\sum_a\int_{\mathbf{r}} G_{cl}(\mathbf{r},\mathbf{r}_a,\omega)G_{cl}(\mathbf{r}_a,\mathbf{r},\omega). \quad (59)$$

Using the Landau-level basis (N—the Landau-level number, $\mathbf{k}$—quasimomentum), see Appendix A and Ref. 18, and its completeness

$$G_{cl}(\mathbf{r},\mathbf{r},\omega) = \sum_{Nk}\frac{|\psi_{N\mathbf{k}}(\mathbf{r})|^2}{i\omega + Nh}, \quad (60)$$

and it simplifies to

$$R(\omega) = \sum_{N\mathbf{k}} \frac{1}{i\omega + Nh} - U_0 \frac{h}{2\pi} \Pi_{\mathbf{q}=\mathbf{0},\boldsymbol{\omega}} \sum_a \sum_{N\mathbf{k}} \frac{|\psi_{N\mathbf{k}}(\mathbf{r}_a)|^2}{(i\omega+Nh)^2}. \tag{61}$$

The ground-state energy is obtained from the LLL ($N=0$) contribution to the resolvent

$$\begin{aligned} R(\omega) &= \frac{h}{2\pi}\frac{1}{i\omega}\left[1 - U_0\Pi_{\mathbf{0},\omega}\sum_a\sum_{\mathbf{k}}\frac{|\psi_{0\mathbf{k}}(\mathbf{r}_a)|^2}{i\omega}\right] \\ &= \frac{h}{2\pi}\frac{1}{i\omega}\left[1 - U_0\Pi_{\mathbf{0},\boldsymbol{\omega}}\sum_a g_{cl}^{\mathrm{LLL}}(\mathbf{r}_a,\mathbf{r}_a,\omega)\right] \\ &= \frac{h}{2\pi}\frac{1}{i\omega}\left(1 - \frac{U_0\rho_p h}{2\pi i\omega}\Pi_{\mathbf{0},\boldsymbol{\omega}}\right). \end{aligned} \tag{62}$$

For the magnetic field considered to obey $h>\rho_p$, it originates from the LLL contribution to $\Pi_{\mathbf{q}=\mathbf{0},\boldsymbol{\omega}}$,

$$R(\omega) = \frac{h}{2\pi}\frac{1}{i\omega}\left(1 - \frac{2\pi U_0\rho_p h\frac{1}{i\omega}}{1+2\pi U_0\rho_p h\frac{1}{i\omega}}\right) = \frac{h}{2\pi}\frac{1}{i\omega+u_0}, \tag{63}$$

where

$$u_0 = 2\pi U_0\rho_p h \equiv uh \tag{64}$$

is the lowest eigenvalue of $\hat{H}_p$. One observes that it is proportional both to the pinning strength and magnetic field. This should be contrasted with the shift of the ground-state energy in the absence of the magnetic field for the same potential which is finite: $u_0=u$. Our derivation is valid only for magnetic fields $h>\rho_p$ to satisfy the commensurability condition and therefore the limit $h\to 0$ cannot be taken. Physically one notes that the system takes advantage of zeros of the wave function created by magnetic field to avoid the increase in ground-state energy due to repelling delta potential barriers.

D. Final expression of the Green's function of $\hat{H}$

To determine the operator $\hat{H}$ of the bifurcation point expansion, Eq. (15), one subtracts the constant u_0 of Eq. (64) from $\hat{H}_p$: $\hat{H}=\hat{H}_p-u_0$. In the ω space such a transformation is equivalent to a shift of frequency by the imaginary number iu_0 in the GF

$$G(\mathbf{r},\mathbf{r}',\omega) \to G(\mathbf{r},\mathbf{r}',\omega+iu_0). \tag{65}$$

Consequently the Eq. (56) can be written as

$$\begin{aligned} G(\mathbf{r},\mathbf{r}',\omega) &= G_{cl}(\mathbf{r},\mathbf{r}',\omega+iu_0) \\ &\quad - \frac{U_0}{S_{\mathrm{BZ}}}\sum_{a,b}\int_{\mathbf{q}} e^{i\mathbf{q}\cdot(\mathbf{r}_b-\mathbf{r}_a)}\Pi_{\mathbf{q},\omega+iu_0}G_{cl}(\mathbf{r},\mathbf{r}_a,\omega \\ &\quad + iu_0)G_{cl}(\mathbf{r}_b,\mathbf{r}',\omega+iu_0). \end{aligned} \tag{66}$$

This explicit expression for the Green's function allows calculation of any transport coefficients including electric conductivity. Note that the quantity $G_{cl}(\mathbf{r}_b,\mathbf{r}',\omega+iu_0)$ should be defined as an analytic continuation of the clean GF, since, as explained above, the spectrum of the "shifted" clean Hamiltonian has negative eigenvalues, However the full GF has positive spectrum and is well defined.

V. ac CONDUCTIVITY OF THE VORTEX LATTICE SLIDING OVER PERIODIC PINNING ARRAY

Returning to the ac conductivity Eq. (33), we substitute the GF of the previous section. It can be divided into two contributions

$$\sigma_s(\omega) = \sigma_I(\omega) + \sigma_{II}(\omega), \tag{67}$$

where

$$\sigma_I(\omega) = -\frac{2}{L_xL_y}\int_{\mathbf{r}\mathbf{r}'} y'\phi(\mathbf{r}')[D_y\phi(\mathbf{r})]^* G_{cl}(\mathbf{r},\mathbf{r}',\omega+iu_0), \tag{68}$$

$$\sigma_{II}(\omega) = \frac{2U_0}{S_{\mathrm{BZ}}}\sum_{a,b}\Sigma_a^1(\omega)\Sigma_b^2(\omega)\int_{\mathbf{q}} e^{i\mathbf{q}(\mathbf{r}_b-\mathbf{r}_a)}\Pi_{\mathbf{q},\omega+iu_0}. \tag{69}$$

Here we defined

$$\Sigma_a^1(\omega) = \frac{1}{\sqrt{L_xL_y}}\int_{\mathbf{r}}[D_y\phi(\mathbf{r})]^* G_{cl}(\mathbf{r},\mathbf{r}_a,\omega+iu_0), \tag{70}$$

$$\Sigma_b^2(\omega) = \frac{1}{\sqrt{L_xL_y}}\int_{\mathbf{r}'} y'\phi(\mathbf{r}')G_{cl}(\mathbf{r}_b,\mathbf{r}',\omega+iu_0). \tag{71}$$

The first part is the flux flow conductivity $\sigma_I=\sigma_{FF}$ in the absence of pinning while the second term vanishes in this limit. We therefore first calculate σ_{FF}.

A. Flux flow ac conductivity in the clean system

Substituting the clean ($u=0$) retarded GF and using the Abrikosov wave function expressed in Appendix A in terms of the Landau harmonics $\varphi_N(\mathbf{r})$, the expression of Eq. (68) includes

$$y'\varphi_0(\mathbf{r}') = \frac{1}{\sqrt{2h}}\varphi_1(\mathbf{r}') + \varphi', \tag{72}$$

where the function φ' belongs to the LLL and therefore do not contribute to the average current. The integration over $\mathbf{r}'$ results in

$$\begin{aligned} \sigma_{FF}(\omega) = 3^{1/8}4\pi\frac{a_h}{\beta_A}\int_{t=0} e^{-i\omega t}C\frac{h-2/\eta}{(2/\eta+h)^2}\sum_l e^{i\pi l^2/2} \\ \times\exp\left[-\frac{\pi^2(2l+1)^2}{2a^2h} - \frac{i\pi l}{2}\right] \\ \times\left\langle\left[y - \frac{\pi}{ah}(2l+1)\right](D_y\varphi_0)^*\right. \end{aligned}$$

$$\times \exp\left[-\frac{i\pi}{a}(2l+1)(x-iy)-\frac{h}{2}y(ix+y)\right]\Bigg\rangle_r, \tag{73}$$

where $C(t)$ and $\eta(t)$ are given in Eq. (38). Substituting $D_y\varphi_0$ from Appendix A, the average over r is performed

$$\sigma_{FF}(\omega)=2\pi\frac{a_h}{\beta_A}\int_{t=0}^{\infty}dte^{-i\omega t}C(t)\frac{2/\eta(t)-h}{[2/\eta(t)+h]^2}=\frac{a_h}{\beta_A}\frac{1}{i\omega+h}, \tag{74}$$

which also can be directly computed using the Landau-level basis.

To make the resulting expression more transparent physically it may be rewritten in terms of dimensional parameters (recalling that this is just the superconducting component)

$$\sigma_{FF}(\omega)=\frac{c^2\gamma}{8\pi\kappa^2\beta_A}\frac{H_{c2}(T)-B}{i\omega t_{\rm GL}H_{c2}+B}, \tag{75}$$

where $H_{c2}(T)=H_{c2}(1-T/T_c)$. There exists a well-defined limit $\omega\to 0$ which coincides with the dc conductivity. For $\omega=0$ the result

$$\sigma_{FF}=\frac{c^2\gamma}{8\pi\kappa^2\beta_A}\frac{H_{c2}(T)-B}{B}=\sigma_n[H_{c2}(T)/B-1] \tag{76}$$

is well known[15] and consistent with the Bardeen-Stephen law derived in the London limit (well-separated vortex cores). Indeed, adding the normal component of the conductivity one obtains

$$\sigma=\sigma_n+\sigma_{FF}=\sigma_n H_{c2}(T)/B. \tag{77}$$

The dependence of the flux-flow conductivity on frequency is very weak for $\omega \ll t_{\rm GL}^{-1}\sim 10^{13}$ Hz, even for low-T_c materials. This is not the case in the presence of strong pinning when electromagnetic shaking even at low frequencies leads to depinning and hence strong increase in resistivity.

B. ac conductivity in the presence of pinning

In the presence of pinning the first contribution to the conductivity, Eq. (68) is renormalized into

$$\sigma_I(\omega)=\frac{a_h}{\beta_A}\frac{1}{i\omega+h-u_0}. \tag{78}$$

Now we consider the second contribution to conductivity Eq. (67). The Gaussian integration over r in the first integral of Eq. (70) gives

$$\begin{aligned}\Sigma_a^1(\omega)=&\frac{4\pi 3^{1/8}}{\sqrt{L_xL_y}}\left(\frac{a_h}{\beta_A}\right)^{1/2}\frac{h}{i\omega+h-u_0}\sum_m e^{-i\pi m^2/2}\\&\times\left[y_a-\frac{\pi(2m+1)}{ah}\right]\\&\times\exp\left\{\frac{i\pi m}{2}-\frac{\pi^2(2m+1)^2}{2a^2h}\right.\\&\left.+\left[\frac{\pi(2m+1)}{a}-\frac{hy_a}{2}\right](y_a-ix_a)\right\}\\&=-\left(\frac{ha_h}{2\beta_A}\right)^{1/2}\frac{\varphi_1^*(r_a)}{\sqrt{L_xL_y}}\frac{1}{i\omega+h-u_0}.\end{aligned} \tag{79}$$

Similarly integration over $\mathbf{r}'$ results in

$$\Sigma_b^2(\omega)=-h\Sigma_b^{1*}(-\omega), \tag{80}$$

so that

$$\sigma_{II}(\omega)=\frac{a_h}{\beta_A}\frac{|\varphi_1(\mathbf{r}=0)|^2}{L_xL_yS_{\rm BZ}}\frac{U_0}{(i\omega+h-u_0)^2}\int_{\mathbf{q}}\sum_{a,b}e^{i\mathbf{q}\cdot(\mathbf{r}_b-\mathbf{r}_a)}\Pi_{\mathbf{q},\omega+iu_0}. \tag{81}$$

Performing the double lattice sum,

$$\sum_{a,b}e^{i\mathbf{q}\cdot(\mathbf{r}_b-\mathbf{r}_a)}=N_p^2\sum_{\mathbf{K}}\delta_{\mathbf{q},\mathbf{K}}, \tag{82}$$

and recalling that q is restricted to the first BZ, the resulting reciprocal-lattice sum reduces to the single term with $\mathbf{q}=\mathbf{K}=\mathbf{0}$ so that

$$\sigma_{II}(\omega)=\frac{a_h}{\beta_A}\frac{U_0\rho_p|\varphi_1(0)|^2}{(i\omega+h-u_0)^2}\Pi_{\mathbf{q}=\mathbf{0},\omega+iu}. \tag{83}$$

An explicit expression for the polarization kernel, Eq. (54), can be obtained by using the results derived in Appendix B, i.e.,

$$\begin{aligned}\int_{\mathbf{q}}\Pi_{\mathbf{q},\omega+iu_0}&=\int_{\mathbf{q}}\frac{1}{1+U_0\rho_p\tilde{g}_{\mathbf{q}=\mathbf{0},\omega+iu_0}},\\&\approx\frac{S_{\rm BZ}}{1+u\left[\frac{h}{i\omega-u_0}+\Theta\left(1+i\frac{\omega}{h}-u\right)\right]},\end{aligned} \tag{84}$$

where

$$\Theta(X)=\log\left(\frac{K_{\max}^2}{2h}\right)-\psi(X). \tag{85}$$

Here ψ is the digamma function and $K_{\max}=2\pi/w$ serves as an ultraviolet cutoff, which is determined by the lateral size, w, size of a pinning center. Finally the second term is

$$\sigma_{II}(\omega)=\frac{a_h}{\beta_A}\frac{3.75U_0\rho_p h}{(i\omega+h-u_0)^2\left[\frac{i\omega}{i\omega-u_0}+u\Theta\left(1+i\frac{\omega}{h}-u\right)\right]}, \tag{86}$$

where we have used $\varphi_1(\mathbf{r}=0)=1.373(1+i)h^{1/2}$ from Appendix A and Eq. (52).

A large positive denominator suppresses the second contribution to the conductivity σ_{II}, Eq. (81), due to "screening" of the pinning potential by excitations with high LL quantum numbers N. In the presence of thermal (or quantum) fluctuations (e.g., close to the vortex lattice melting point), when the energy scale of the vortex lattice shear fluctuations[19] becomes comparable to the pinning energy scale u, this denominator can be reduced significantly resulting in large enhancement of the pinning conductivity.

Combining the two contribution of the previous section, one expresses the ac conductivity at frequency ω as a function of two dimensionless parameters $u=2\pi U_0\rho_p$ (pinning strength) and h,

$$\sigma_s(\omega)=\frac{a_h}{\beta_A}\frac{1}{i\omega+h\Delta}\times\left\{1+\frac{0.6u}{\left(i\frac{\omega}{h}+\Delta\right)\left[\frac{i\omega}{i\omega-uh}+u\Theta\left(i\frac{\omega}{h}+\Delta\right)\right]}\right\}, \tag{87}$$

where

$$\Delta=1-u \tag{88}$$

is the distance from the "pinning-depinning line." It is important to note that for $\Delta<0$ the linear-response approximation is invalid. Physically this means that the flux flow state does not exists and the I-V curve becomes nonlinear. The reason is that there exists (assuming no thermal fluctuations on the mesoscopic scale) a nonzero critical (threshold) current. Vortices are effectively pinned and as a consequence electric field cannot penetrate the superconductor.

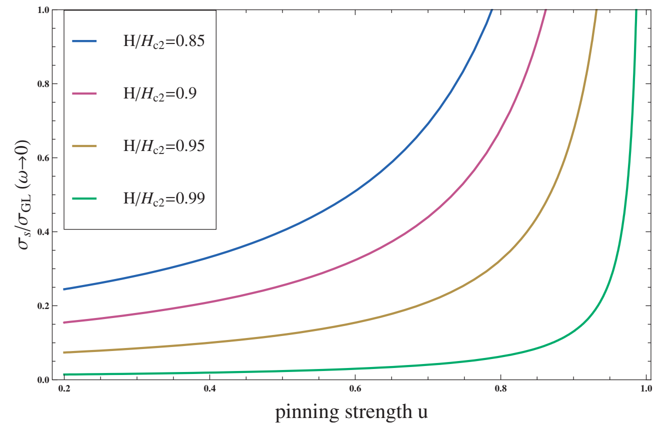

FIG. 2. (Color) Conductivity at $\omega\to0$ as function of the pinning strength u for magnetic field in the $h=0.85-0.99$ range. When the pinning strength approaches the critical, $u=u_c$, the conductivity diverges.

In Fig. 2 the dependence in the $\omega\to0$ limit on the pinning strength u for magnetic fields in the $h=0.85-0.99$ range is presented. One observes that when the pinning strength approaches the critical value, $u=1$, the conductivity diverges. In Fig. 3 the dependence of the real [dissipation, Figs. 3(a) and 3(c)] and the imaginary [inductive, Figs. 3(b) and 3(d)] parts of the ac conductivity as function of u at two values of magnetic fields close to $H_{c2}(T)$ [$h=0.95$ in Fig. 3(a) and $h=0.99$ in Fig. 3(b)] is shown.

Returning to physical units, one obtains

$$\sigma(\omega)=\frac{c^2\gamma}{2\kappa^2\beta_A}\frac{H_{c2}(T)-B}{i\gamma\omega\Phi_0+4\pi B(1-u)}\left\{1+\frac{0.6u}{\left(i\frac{\gamma\Phi_0\omega}{4\pi B}+1-u\right)\left[\frac{i\gamma\Phi_0\omega}{i\gamma\Phi_0\omega-4\pi uB}+u\Theta\left(i\frac{\gamma\Phi_0\omega}{4\pi B}+1-u\right)\right]}\right\}, \tag{89}$$

where $H_{c2}(T)=H_{c2}(1-T/T_c)$ within linear approximation for the coefficient a' of the Ginzburg-Landau energy, Eq. (3). Note that, as mentioned above, use of the a_h expansion restricts the range of frequencies since the correction to resistivity, Eq. (32), should not exceed ρ_n. In the next section we analyze several simple cases which explain the main features of the conductivity shown in Figs. 2 and 3 and, in particular, the transition to the pinned state.

VI. CRITICAL BEHAVIOR NEAR THE DEPINNING LINE

A. Criticality for small frequencies

When pinning is present the limit $\omega\to0$ in the expression for the ac conductivity is nonzero and exhibits criticality features at the continuous transition

$$h=u_0. \tag{90}$$

Approaching the line $u_0=h-\Delta$ for small Δ the first contribution to the conductivity, Eq. (87), diverges

$$\sigma(\omega\to0)=\frac{a_h}{\beta_A}\frac{1}{\Delta}+\frac{a_h}{\beta_A}\frac{1}{\Delta^2}\frac{0.6}{\Theta(\Delta)}\approx\frac{a_h}{\beta_A}\frac{1.6}{\Delta}. \tag{91}$$

The depinning line, Eq. (90), determines the critical pinning strength since according to Eq. (64)

$$h=u_0=2\pi U_0^c\rho_p h\to U_0^c=\frac{1}{2\pi\rho_p}. \tag{92}$$

which, in terms of dimensional parameters, reads

$$\frac{2\pi^2N_pw^2\varepsilon}{L_xL_yT_c}=1. \tag{93}$$

Therefore the pinning strength is only factor determining the transition into the pinned state. The critical value is independent of the magnetic induction. To conclude the conductivity diverges as a power $\Delta^{-\nu(z-1)}$ with critical exponent $\nu(z-1)=1$, hence superconductivity is recovered. This means that

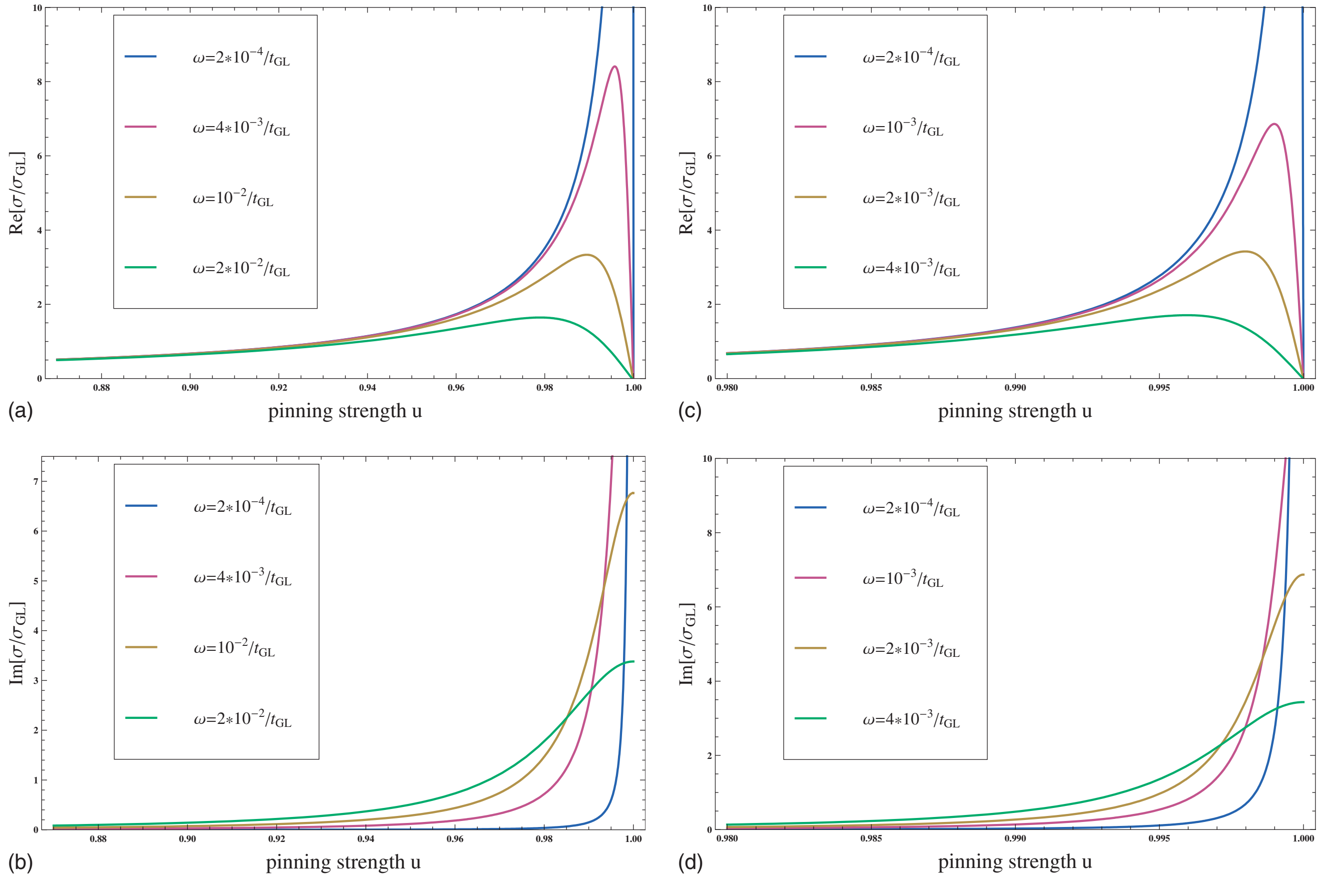

FIG. 3. (Color) ac conductivity for various frequencies as function of the pinning strength u for magnetic field close to H_{c2}: $h=0.95$ in (a) and (b) and $h=0.99$ in (c) and (d). The real part [dissipation, (a) and (c)] and the imaginary part [inductive, (b) and (d)] are shown.

the vortex lattice is pinned and electric field cannot penetrate the superconductor despite persistent current flow in it at least when the current is not large. This is consistent with results in the London limit for both a 2D system with point-like defects and a three-dimensional system with columnar defects[2] (in the absence of thermal fluctuations on the mesoscopic scale).

B. Dependence of the ac conductivity on frequency on the depinning line

It can be readily seen by analyzing Eqs. (78) and (86) that on the depinning line $\Delta=0$ the ac conductivity of Eq. (87) simplifies to

$$\sigma(\omega)=\frac{a_h}{\beta_A}\frac{1}{i\omega}\left\{1+\frac{0.6}{i\dfrac{\omega}{h}\left[\dfrac{i\omega}{i\omega-h}+\Theta\left(i\dfrac{\omega}{h}\right)\right]}\right\}, \tag{94}$$

and for $\omega \ll h$ leads to the same dynamic critical exponent

$$\sigma(\omega)=\frac{a_h}{\beta_A}\frac{1.6}{i\omega}. \tag{95}$$

C. Subcritical pinning strength

When $u \ll 1$ the expression (87) can be simplified by expansion in this parameter ($\Delta=1$),

$$\sigma(\omega)=\frac{a_h}{\beta_A}\frac{1}{i\omega+h}\times\left\{1+\frac{0.6u}{\left(i\dfrac{\omega}{h}+1\right)\left[\dfrac{i\omega}{i\omega-uh}+u\Theta\left(i\dfrac{\omega}{h}+1\right)\right]}\right\}. \tag{96}$$

Let us consider two cases. The simplest case is for very large frequencies or very small fields, $\omega \gg h$. In this case the conductivity simplifies significantly

$$\sigma(\omega)=\frac{a_h}{\beta_A}\frac{1}{i\omega}. \tag{97}$$

It is largely inductive and pinning does not influence it. In the opposite case of very large fields or relatively small frequencies, $h \gg \omega$, considered next disorder is important.

In the opposite case of very large fields or relatively small frequencies, $\omega \ll h$, Eq. (96) in this case simplifies into

$$\sigma(\omega)=\frac{a_h}{\beta_A}\frac{1}{h}\frac{i\omega-[\Theta(1)+0.6]u^2h}{i\omega-\Theta(1)u^2h}. \tag{98}$$

There are two cases. For $\omega \ll hu^2$ the system becomes purely dissipative

$$\sigma(\omega) = \frac{a_h}{\beta_A}\frac{1}{h}\frac{\Theta(1)+0.6}{\Theta(1)}. \tag{99}$$

In the intermediate case $hu^2 \ll \omega \ll h$

$$\sigma(\omega) = \frac{a_h}{\beta_A}\frac{1}{h}. \tag{100}$$

VII. SUMMARY

To summarize we have extended the theory of electromagnetic response of a type-II superconductor in magnetic field with a periodic array of pinning centers beyond the London approximation (employed commonly in the literature) using the Ginzburg-Landau approach. The ac conductivity as a function of magnetic field and frequency, see Eq. (89) and Figs. 2 and 3, is calculated analytically and can be applied to an experimentally important regime of strong magnetic fields when the London approximation is inapplicable. This is the main result of the present paper. The analytic solution was obtained by the Lippmann-Schwinger method not far from the $H_{c2}(T)$ line for a 2D Abrikosov lattice of vortices is commensurate with the short-range pinning array.

It is predicted that in strong magnetic fields the short-range pinning can effectively influence the vortex dynamics due to long-range correlations of the superconducting order parameter. In particular, a magnetic field independent critical pinning strength

$$U_0^c = \frac{\pi w^2 \varepsilon}{T_c} = \frac{l^2}{2\pi}, \tag{101}$$

was found at which the conductivity at low frequencies diverges as a power[2] $\sigma \propto (U_0 - U_0^c)^{-\nu(z-1)}$ with the critical exponent $\nu(z-1)=1$. Here l is the distance between pinning sites of (energy) depth ε and width w. For columnar defects due to irradiation $\varepsilon \sim 2T_c$, $w \sim \xi(T)$, making this condition $l < 2\pi\xi(T)$. At criticality the conductivity diverges Using an accepted 2D value of $\nu=1$,[2] this implies $z=2$. The ac conductivity on the depinning line diverges as $i\omega^{-1}$. Below U_0^c and sufficiently low frequencies there exists a sliding Abrikosov lattice, which moves nearly uniformly. This sliding lattice behavior, which does not exists in the single vortex-pinning regime, becomes possible due to strong interactions between vortices when they outnumber the columnar defects. Physically it is caused by "liberation" of the temporarily trapped vortices by their freely moving neighbors.

Finally, let us qualitatively contrast the results of two phenomenological approaches to the problem, the TDGL method adopted in the present paper and the London approximation approach used in previous numerical simulation.[11,20] The former is valid for fields much larger than H_{c1} while the later is valid far below H_{c2}. In strongly type-II superconductors the applicability ranges might overlap. The difference is more pronounced in the case of short-range pinning considered in the present paper. Indeed this type of pinning is very ineffective within the London approach since it assumes that a vortex is a "pointlike" 2D particle and consequently its velocity tracks the pinning potential landscape only at pinning centers. In contrast, within the GL approach the mixed state is described by a wave function. It senses pinning potential even when the size of the pin is smaller than $\xi(T)$. Formally it is very similar to influence of the deltalike potential on wave function in quantum mechanics, as we have used while calculating Green's functions in Sec. III.

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APPENDIX A: ABRIKOSOV FUNCTIONS COMMENSURATE WITH A HEXAGONAL ARRAY OF PINNING CENTERS

The first two Landau harmonics in dimensionless units are

$$\varphi_0 = 3^{1/8}\sqrt{h}\sum_l e^{i\pi l^2/2} \exp\left\{ i\left[-\frac{h}{2}xy + \frac{\pi(2l+1)}{a}\left(x-\frac{a}{4}\right)\right] - \frac{h}{2}\left[y - \frac{\pi(2l+1)}{ah}\right]^2\right\}, \tag{A1}$$

$$\varphi_1 = 2^{1/2}3^{1/8}h\sum_l \left[y - \frac{\pi(2l+1)}{ah}\right] e^{i\pi l^2/2}\exp\left\{ i\left[-\frac{h}{2}xy + \frac{\pi(2l+1)}{a}\left(x-\frac{a}{4}\right)\right] - \frac{h}{2}\left[y - \frac{\pi(2l+1)}{ah}\right]^2\right\}, \tag{A2}$$

where the dimensionless Abrikosov lattice spacing $a=a_\Delta/\xi$. In particular, at origin $\varphi_0(0)=0$ while

$$\varphi_1(0) = -3^{3/8}\sqrt{\pi h/2}\sum_l (2l+1) \times \exp\left\{\frac{\pi}{4}\left[i(2l^2-2l-1) - \frac{\sqrt{3}}{2}(2l+1)^2\right]\right\}.$$

We also need the covariant derivative of the LLL,

$$D_y\varphi_0 = \left(\frac{\partial}{\partial y} + i\frac{h}{2}x\right)\varphi_0 = -\sqrt{\frac{h}{2}}\varphi_1. \tag{A3}$$

APPENDIX B: FOURIER TRANSFORM OF THE GAUGE INVARIANT PART OF THE GREEN'S FUNCTION OF THE CLEAN SYSTEM

In the t space the inverse Fourier transform with respect to position is

$$g^{cl}_{\mathbf{Q}}(t) = \int_{\mathbf{r}} g^{cl}(\mathbf{r},t)\exp(-i\mathbf{Q}\cdot\mathbf{r}) = \int_{\mathbf{r}} C(t)\exp\left[-\frac{r^2}{2\eta(t)}\right]\exp(-i\mathbf{Q}\cdot\mathbf{r}) = 2\pi\eta(t)C(t)\exp\left[-\frac{\eta(t)Q^2}{2}\right], \quad (B1)$$

where $\mathbf{Q}=\mathbf{q}+\mathbf{K}$, with $\mathbf{q}$ belonging to the first Brillouin zone and $\mathbf{K}$ runs over the reciprocal lattice of the pinning centers. This is transformed into the ω space $g^{cl}_{Q,\boldsymbol{\omega}}=\int_t^{e^{-i\omega t}} g^{cl}_{\mathbf{Q}}(t)$. The quantity appearing in the expression for polarization kernel, Eq. (54), has a form

$$\tilde{g}_{\mathbf{q},\boldsymbol{\omega}} = 2\pi\sum_{\mathbf{K}}\int_t e^{-i\omega t}\eta(t)C(t)\exp\left[-\frac{\eta(t)(\mathbf{q}+\mathbf{K})^2}{2}\right]. \quad (B2)$$

It is useful to divide the integrand into the well-known LLL part,[21] (which diverges when $\omega=0$) and the rest as (higher Landau levels, HLL)

$$\tilde{g}_{\mathbf{q},\boldsymbol{\omega}} = S_{BZ}(g^{\mathrm{HLL}}(\mathbf{Q}) + g^{\mathrm{LLL}}(\mathbf{Q})), \quad (B3)$$

$$g^{\mathrm{HLL}} = 2\int_{t=0}^{\infty} e^{-i\omega t}\sum_{\mathbf{K}}\left\{\frac{\exp\left[-\frac{\mathbf{Q}^2}{h}\tanh\left(\frac{ht}{2}\right)\right]}{1+\exp(-ht)} - \exp\left(-\frac{\mathbf{Q}^2}{h}\right)\right\}, \quad (B4)$$

$$g^{\mathrm{LLL}} = 2\sum_{\mathbf{K}}\exp\left(-\frac{\mathbf{Q}^2}{h}\right)\int_{t=0}^{\infty} e^{-i\omega t} = \frac{2}{i\omega}\sum_{\mathbf{K}}\exp\left(-\frac{\mathbf{Q}^2}{h}\right). \quad (B5)$$

For the large values of the filling factor $f=s_1 s_2$ characterizing the pinning arrays under study any value of q within the first BZ of the pinning-center lattice is small on the scale of the inverse magnetic length so that we can take $\mathbf{q}=\mathbf{0}$ in Eqs. (B4) and (B5). Under the same assumption the sum over reciprocal lattice can be approximated into an integral. Thus, for the LLL part we have

$$g^{\mathrm{LLL}} = \frac{2}{i\omega}\sum_{\mathbf{K}}\exp\left(-\frac{\mathbf{K}^2}{h}\right) = \frac{2}{i\omega S_{\mathrm{BZ}}}\int_{\mathbf{K}}\exp\left(-\frac{\mathbf{K}^2}{h}\right) = \frac{2\pi h}{S_{\mathrm{BZ}}}\frac{1}{i\omega}. \quad (B6)$$

Similarly, the HLL part becomes

$$g^{\mathrm{HLL}} = \int_{t=0}^{\infty} e^{-i\omega t}\frac{f}{2\pi h}\int_{\mathbf{K}}\left\{\frac{e^{ht/2}}{\cosh(ht/2)} \times\exp\left[-\frac{\mathbf{K}^2}{h}\tanh\left(\frac{ht}{2}\right)\right] - 2\exp\left(-\frac{\mathbf{K}^2}{h}\right)\right\}, \quad (B7)$$

which reduces to

$$g^{\mathrm{HLL}} = \frac{f}{2\pi h}\int_{t=0}^{\infty} e^{-i\omega t}\int_0^{K^2_{\mathrm{max}}} dK^2 \times\left\{\frac{\exp\left[-\frac{K^2}{h}\tanh\left(\frac{ht}{2}\right)\right]}{e^{-ht}+1} - \exp\left(-\frac{K^2}{h}\right)\right\} \quad (B8)$$

$$= -\frac{f}{2\pi}\int_{t=0}^{\infty}[1+(e^{-i\omega t}-1)]I(t) = g_1^{\mathrm{HLL}} + g_2^{\mathrm{HLL}}, \quad (B9)$$

where

$$I(t) = \frac{\exp\left[-\frac{K^2_{\mathrm{max}}}{h}\tanh\left(\frac{ht}{2}\right)\right]-1}{1-e^{-ht}} + 1 - \exp\left(-\frac{K^2_{\mathrm{max}}}{h}\right), \quad (B10)$$

and $K_{\mathrm{max}}=2\pi/w$ is the ultraviolet cutoff of order of the inverse width of columnar defect which in turn is of order larger than the inverse coherence length. Dependence on the cutoff is logarithmic and is obtained from the first term,

$$g_1^{\mathrm{HLL}} = \frac{f}{2\pi h}\left[\log\left(\frac{K^2_{\mathrm{max}}}{2h}\right) + \gamma_E + O(K^{-1}_{\mathrm{max}})\right], \quad (B11)$$

where γ_E is the Euler constant. The second term is calculated at the $K^2_{\mathrm{max}}\to\infty$ limit directly

$$g_2^{\mathrm{HLL}} = \frac{f}{2\pi h}\left[-\gamma_E - \psi\left(i\frac{\omega}{h}+1\right)\right]. \quad (B12)$$

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